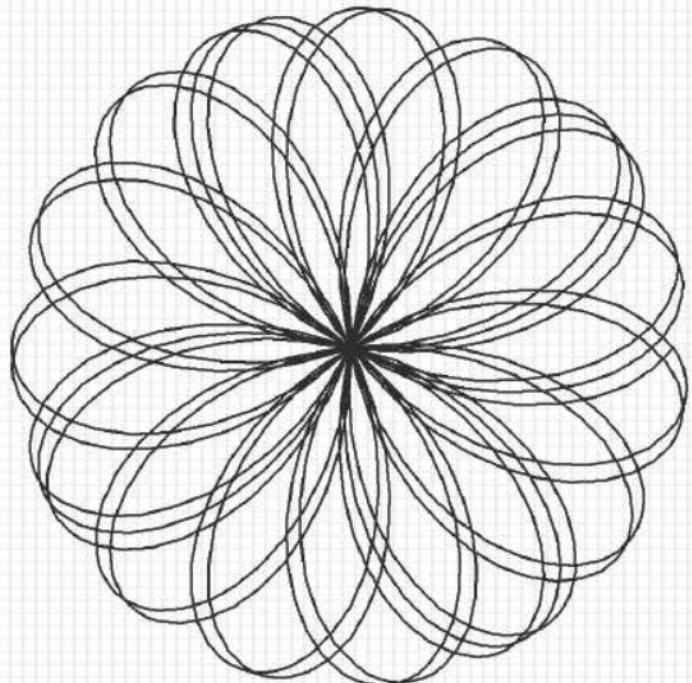


10.6: The Calculus of Polar Curves - Slopes and Arc Length



$$r = 2 \sin(2.15\theta)$$
$$0 \leq \theta \leq 16\pi$$

To find the slope of a polar curve at a given point:

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(y)}{\frac{d}{d\theta}(x)} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{r(-\sin \theta) + \cos \theta \frac{dr}{d\theta}}$$

The one to remember is:

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)}$$



Example: $r = 1 - \cos\theta$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{d\theta}(r \sin\theta)}{\frac{d}{d\theta}(r \cos\theta)} = \frac{\frac{d}{d\theta}((1-\cos\theta) \sin\theta)}{\frac{d}{d\theta}((1-\cos\theta) \cos\theta)} \\ &= \boxed{\frac{(1-\cos\theta)\cos\theta + \sin\theta(-\sin\theta)}{(1-\cos\theta)(-\sin\theta) + \cos\theta(\cos\theta)}}\end{aligned}$$



Ex: Find the slope of the tangent line to the circle $r = 4\cos\theta$ when $\theta = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{4 \frac{d}{d\theta} (\cancel{4\cos\theta} \cdot \sin\theta)}{4 \frac{d}{d\theta} (\cancel{4\cos\theta} \cdot \cos\theta)}$$

$$\frac{dy}{dx} = \frac{\cos\theta(\cos\theta) + \sin\theta(-\sin\theta)}{2\cos\theta \cdot \sin\theta}$$

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{4}} = \frac{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}} = 0$$

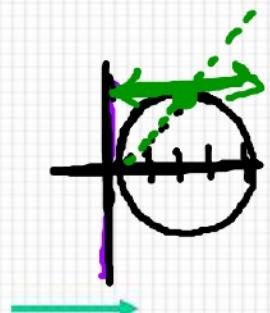
HORIZ. TANGENT $\Rightarrow y = k$

$$y = r\sin\theta$$

$$y = (4\cos\theta)\sin\theta$$

$$y\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 2$$

$$\boxed{y=2}$$



Horizontal Tangent:

$$\frac{dy}{d\theta} = 0 \quad \text{and} \quad \frac{dx}{d\theta} \neq 0$$

Vertical Tangent:

$$\frac{dx}{d\theta} = 0 \quad \text{and} \quad \frac{dy}{d\theta} \neq 0$$



Ex: Find the equations of all horizontal and vertical tangent lines to
 $r = 1 - \cos\theta$.

Horizontal: $\frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = (1-\cos\theta)\cos\theta + \sin\theta(-\sin\theta)$$

$$0 = \cos\theta - \cos^2\theta + \sin^2\theta$$

$$0 = \cos\theta - \cos^2\theta + 1 - \cos^2\theta$$

$$0 = +2\cos^2\theta - \cos\theta + 1$$

$$0 = (2\cos\theta + 1)(\cos\theta - 1)$$

$$\cos\theta = -\frac{1}{2} \quad \cos\theta = 1$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \theta = \boxed{0}$$

$$y = r\sin\theta = (1-\cos\theta)\sin\theta$$

$$y(0) = (1-1)(0) = 0 \Rightarrow \boxed{y=0}$$

$$y\left(\frac{2\pi}{3}\right) = \left(1 - \frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4} \Rightarrow \boxed{y = \frac{3\sqrt{3}}{4}}$$

$$y\left(\frac{4\pi}{3}\right) = \left(1 - \frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \boxed{y = -\frac{3\sqrt{3}}{4}}$$



Ex: Find the equations of all horizontal and vertical tangent lines to
 $r = 1 - \cos\theta$.

Vertical: $\frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = (\sin\theta)(\cos\theta) + (1-\cos\theta)(-\sin\theta)$$

$$0 = \sin\theta\cos\theta - \sin\theta + \sin\theta\cos\theta$$

$$0 = 2\sin\theta\cos\theta - \sin\theta$$

$$0 = \sin\theta(2\cos\theta - 1)$$

$$\sin\theta = 0 \quad \cos\theta = 1/2$$

$$\theta = \cancel{\pi}, \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$x = r\cos\theta = (1-\cos\theta)\cos\theta$$

$$x(\pi) = (1-1)(1) = -2 \Rightarrow \boxed{x = -2}$$

$$x\left(\frac{\pi}{3}\right) = \left(1 - \frac{1}{2}\right) \cdot \frac{1}{2} = \frac{1}{4} \Rightarrow \boxed{x = 1/4}$$

$$x\left(\frac{5\pi}{3}\right) = \left(1 - \frac{1}{2}\right) \cdot \frac{1}{2}$$



Ex: Find the equations of all horizontal and vertical tangent lines to $r = 1 - \cos\theta$.

What about values that make both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta} = 0$? $\theta = 0$

$$\lim_{\theta \rightarrow 0} \frac{dy}{dx} \Rightarrow \frac{0}{0} \Rightarrow \text{USE L'HOPITAL'S}$$

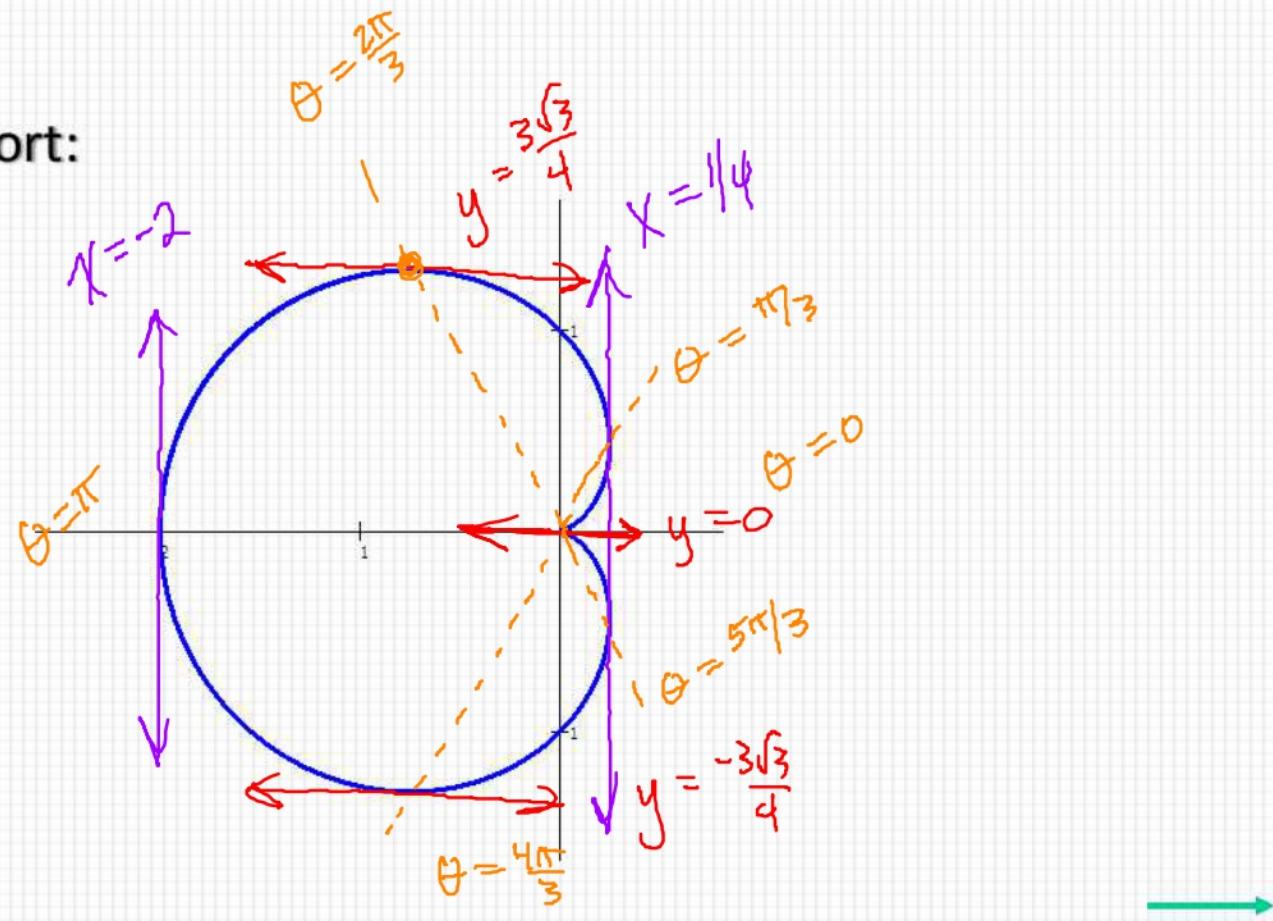
$$\hookrightarrow \lim_{\theta \rightarrow 0} \frac{4\cos\theta(-\sin\theta) + \sin\theta}{2(\cos\theta\cos\theta + \sin\theta\cdot -\sin\theta) - \cos\theta} = \frac{0}{1}$$

∴ HORIZONTAL TANGENT @ $\theta = 0$.



Ex: Find the equations of all horizontal and vertical tangent lines to $r = 1 - \cos\theta$.

Graphical Support:



Ex: Find the equations of the lines tangent to $r = 2 \sin(3\theta)$ on $[0, 2\pi]$ at the pole. (ORIGIN) AT ORIGIN: $\frac{dy}{dx} = \tan\theta$

$$0 = 2 \sin 3\theta$$

$$\sin 3\theta = 0$$

$$(3\theta) = 0, \pi, 2\pi, \dots$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \dots$$

$$\left. \frac{dy}{dx} \right|_{\theta=0} = \tan 0 = 0 \Rightarrow \boxed{y=0}$$

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{3}} = \tan \frac{\pi}{3} = \sqrt{3} \Rightarrow \boxed{y = \sqrt{3}x}$$

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{2\pi}{3}} = \tan \frac{2\pi}{3} = -\sqrt{3} \Rightarrow \boxed{y = -\sqrt{3}x}$$

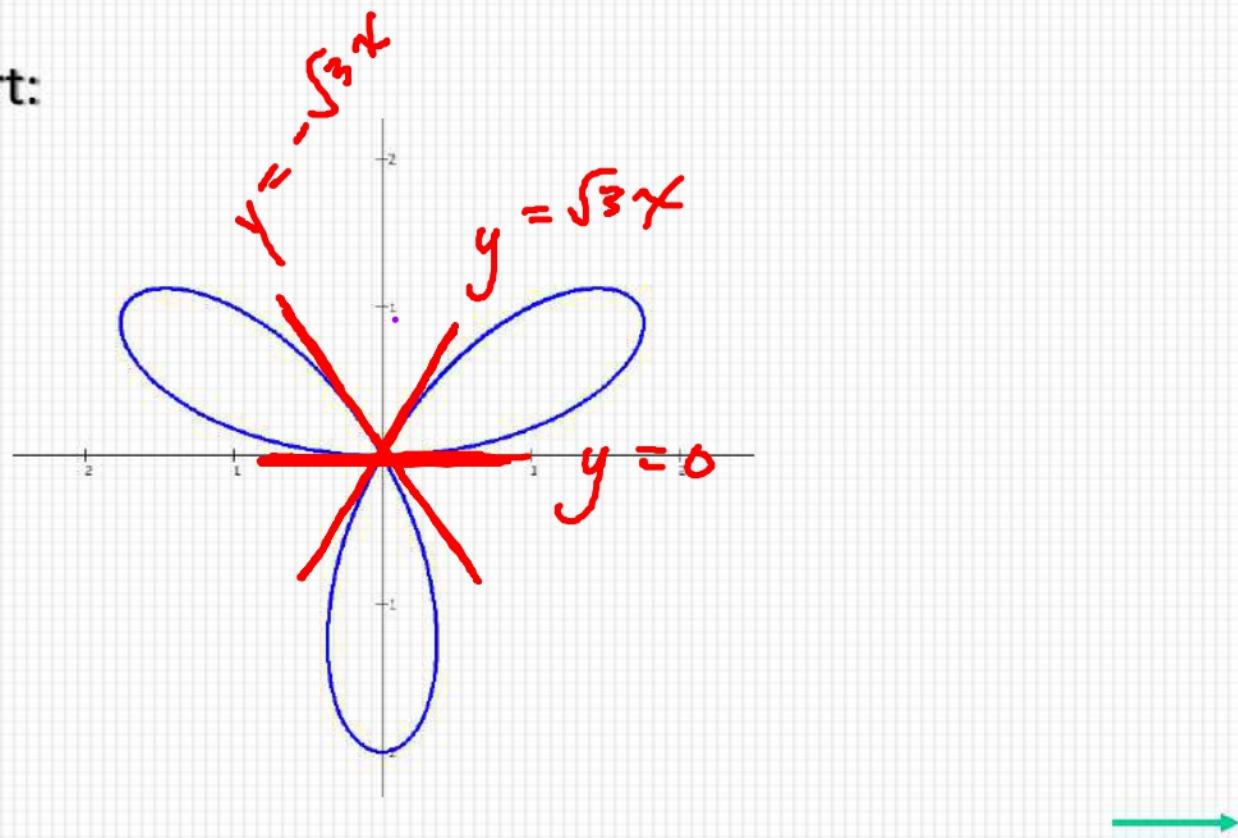
$$\left. \frac{dy}{dx} \right|_{\theta>\pi} = \tan \pi = 0$$

⋮



Ex: Find the equations of the lines tangent to $r = 2 \sin(3\theta)$ on $[0,2\pi]$ at the **pole**.

Graphical support:



To find the length of a curve in parametric mode:

Polar is parametric with parameter θ :

If we do a bunch or simplifying (proof omitted), we get:



Ex: Find the length of the polar curve $r = 1 - \cos\theta$.

Ex: Find the length of the polar curve $r = e^\theta$ from $\theta = 0$ to $\theta = 1$.



Homework:

Section 10.6 WS – Tangents in Polar

Section 10.6 WS – Arc Length in Polar

